

Rigidity of the topological dual of spaces of formal series with respect to product topologies

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Abstract. Even in spaces of formal power series is required a topology in order to legitimate some operations, in particular to compute infinite summations. Many topologies can be exploited for different purposes. Combinatorists and algebraists may think to usual order topologies, or the product topology induced by a discrete coefficient field, or some inverse limit topologies. Analysts will take into account the valued field structure of real or complex numbers. As the main result of this paper we prove that the topological dual spaces of formal power series, relative to the class of product topologies with respect to Hausdorff field topologies on the coefficient field, are all the same, namely the space of polynomials. As a consequence, this kind of rigidity forces linear maps, continuous for any (and then for all) of those topologies, to be defined by very particular infinite matrices similar to row-finite matrices.

Résumé. La validité de certaines opérations, notamment les calculs de sommes de séries, repose sur l'emploi d'une topologie sur l'espace des séries formelles. En Combinatoire ou en Algèbre il est d'usage de considérer les topologies données par une valuation, ou les topologies produits relativement à un corps discret, ou encore des topologies limites projectives. Dès lors que l'Analyse entre en scène il devient inévitable de considérer les valeurs absolues des corps des nombres réels et complexes. Il s'avère en fait que les duals topologiques des espaces de séries, par rapport aux topologies produits relatives aux copies d'un même corps topologique séparé, sont tous isomorphes à l'espace des polynômes. La preuve de ce résultat de rigidité constitue l'apport principal de notre article. L'indépendance du dual topologique relativement au choix parmi ces topologies astreint les opérateurs linéaires, continus pour une (et alors pour toutes les) topologie(s) de notre collection, à apparaître sous la forme de matrices infinies bien particulières qui n'ont qu'un nombre fini d'entrées non nulles sur chaque "ligne".

Keywords: Topological fields, topological duals, product topology, infinite matrices

1 Introduction

Manipulation of formal power series requires some topological notions in order to legitimate some computations. For instance, the usual substitution of a power series in one variable without constant term into another, or the existence of the star operation, related to Möbius inversion, are usually treated using either an order function (a valuation) or, equivalently (while more imprecise), saying that only finitely many terms contribute to the calculation in each degree. In both cases is used (explicitly or not) a topology given by a filtration (and more precisely a kind of Krull topology): the order of some partial sums must increase indefinitely for the sum to be defined (and the operation to be legal). Quite naturally other topologies may

be used: for instance if X is an infinite set, then the completion of the algebra $R\langle X \rangle$ of polynomials in noncommutative variables (where R is a commutative ring with a unit) with respect to the usual filtration (related to the length of a word in the free monoid X^*) is the set of all series with only finitely many non-zero terms for each given length. The sum of the alphabet does not even exist in this completion. In order to take such series into account we must use the product topology (with a discrete R). If some analytical investigations must be performed (such as convergence ray, differential equation, or functional analysis), the discrete topology of $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is no more sufficient; the valued field structure of \mathbb{K} turns to be unavoidable. Other topologies may be used for particular needs.

Given a topology, compatible in a certain sense with algebraic operations, on a space of formal power series with coefficients in a topological field, it can be useful to consider continuous endomorphisms since they commute to infinite sums. Quite amazingly for a very large class of possible topologies (namely product topologies with respect to separated topologies on the base field) it appears that these continuous linear maps may be seen as infinite matrices of a particular kind (each “row” is finitely supported) and that, independently of the topology chosen in the class. In other terms, a linear map can be represented as some “row-finite” matrix if, and only if, it is continuous for all the topologies in the given class (see Section 5). Thus, in order to prove an endomorphism to be continuous for some topology, it suffices to prove this property for the more convenient topology in the class. Similarly, if an endomorphism is given in the form of an infinite matrix (with finitely supported “rows”), then it is known to be continuous for every topology in our large collection, and therefore is able to get through infinite sums. At this stage we notice that the class of topologies is that of product topologies relative to every Hausdorff topologies on the coefficient field (an infinite field has the cardinal number of the power set of its power set of distinct field topologies, see Remark 1).

The explanation of this phenomenon relies on the fact that the topological dual of a given space of formal series is the same for all the topologies in the big class: the space of polynomials. This result, presented in Section 3 and proved in Section 4, is our main theorem (Theorem 5), and we present some of its direct consequences (in particular it explains the “rigidity” of the space of continuous endomorphisms with respect to a change of topology in Section 5).

In this paper we focus on the linear structure of formal series so that we consider any set X rather than a free monoid X^* , and functions rather than formal series.

2 Some notations

Let R be a commutative ring⁽ⁱ⁾ with a unit 1_R . If M and N are two R -modules, then $\text{Hom}_R(M, N)$ is the set of all R -linear maps from M to N . In particular, the *algebraic dual* M^* of M is the R -module $\text{Hom}_R(M, R)$. If X is a set, then $R^{(X)}$ is the R -module of all finitely supported maps from X to R , that is, $f \in R^{(X)}$ if, and only if, the *support* $\text{supp}(f) = \{x \in X : f(x) \neq 0\}$ of f is a finite set. For every $x_0 \in X$, we define the *characteristic function* (or *Dirac mass*)

$$\begin{aligned} \delta_{x_0}: \quad X &\rightarrow R \\ x &\mapsto \begin{cases} 1_R & \text{if } x = x_0, \\ 0_R & \text{otherwise.} \end{cases} \end{aligned} \tag{1}$$

⁽ⁱ⁾ In the remainder, the term “ring” with no exception refers to “commutative ring with a unit”.

Let us introduce the usual *evaluation map*

$$\begin{aligned} \text{ev}: R^{(X)} \otimes_R R^X &\rightarrow R \\ p \otimes f &\mapsto \sum_{x \in X} p(x)f(x). \end{aligned} \quad (2)$$

As usually it is treated as a R -bilinear map $\langle \cdot, \cdot \rangle$, called *dual pairing* (or *canonical bilinear form*, see ? or ?), *i.e.*,

$$\text{ev}(p \otimes f) = \langle p, f \rangle.$$

In particular, for every $x \in X$, $\text{ev}(\delta_x \otimes f) = \langle \delta_x, f \rangle = f(x)$, and then $\pi_x: f \mapsto \langle \delta_x, f \rangle$ is the *projection* of R^X onto $R\delta_x \cong R$. The dual pairing has the obvious properties of non-degeneracy:

1. for every $p \in R^{(X)} \setminus \{0\}$, there is some $f \in R^X$ such that $\langle p, f \rangle \neq 0$,
2. for every $f \in R^X \setminus \{0\}$, there is some $p \in R^{(X)}$ such that $\langle p, f \rangle \neq 0$.

Let E be a set, and \mathcal{F} be a collection of maps $\phi: E \rightarrow E_\phi$, where each E_ϕ is a topological space. The *initial topology* induced by \mathcal{F} on E is the coarsest topology that makes continuous each ϕ . This topology is Hausdorff if, and only if, \mathcal{F} *separates* the elements of E (*i.e.*, for every $x \neq y$ in E , there exists some $\phi \in \mathcal{F}$ such that $\phi(x) \neq \phi(y)$), and E_ϕ is separated for each $\phi \in \mathcal{F}$. Now, if $E = F^X$, where X is a set, and F is a topological space, then the *product topology* (or *topology of simple convergence* or *function topology*) on F^X is the initial topology induced by the canonical projections $\pi_x: f \mapsto f(x)$, $x \in X$, from E onto F . It is characterized by the following property: For every topological space (Y, τ_Y) , the map $f: Y \rightarrow F^X$ is continuous if, and only if, every map $\pi_x \circ f: Y \rightarrow F$ is continuous (see ?).

Let R be a ring. It is said to be a *topological ring* when it is equipped with a topology τ (not necessarily Hausdorff) for which ring operations, $x \mapsto -x$, $(x, y) \mapsto x + y$, $(x, y) \mapsto xy$, are continuous when is considered on $R \times R$ the product topology defined by τ on each factors. A field \mathbb{K} which is also a topological ring is said to be a *topological field* when the map $x \mapsto x^{-1}$ is continuous on $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$ (equipped with the subspace topology).

If R (resp. \mathbb{K}) is a topological ring (resp. topological field), any R -module M (resp. \mathbb{K} -vector space) is said to be a *topological R -module* (resp. a *topological \mathbb{K} -vector space*) if it is equipped with a topology (Hausdorff or not) that makes continuous the module maps $x \mapsto -x$, $(x, y) \mapsto x + y$, $(\lambda, x) \mapsto \lambda x$. Notice that if R (resp. \mathbb{K}) is a topological ring (resp. topological field), and X is any set, then R^X (resp. \mathbb{K}^X) is a topological R -module (resp. topological \mathbb{K} -vector space) – in the obvious way – when endowed with the product topology.

Let R be a topological ring, and M be a topological R -module. Then M' denotes the *topological dual* of M , that is, the sub-module of M^* of continuous linear forms⁽ⁱⁱ⁾.

Summability. Many intermediary results of this papers require the notion, and some properties, of a summable family in a topological ring (which is actually hidden but fundamental for the treatment of formal power series). We recall them without any proof; we freely used them in the sequel, and refer to ? for further information concerning this concept but also topological rings, fields and modules.

⁽ⁱⁱ⁾ Quite obviously, M^* is the topological dual of M , when R and M both have the discrete topology.

Definition 1 Let G be a Hausdorff Abelian group (that is, an Abelian group – in additive notation – with a separated topology such that the group operations, addition and inversion, are continuous), and $(x_i)_{i \in I}$ be a family of elements of G . An element $s \in G$ is the sum of the summable family $(x_i)_{i \in I}$ if, and only if, for each neighborhood V of s there exists a finite subset $J \subseteq I$ such that $\sum_{j \in J} x_j \in V$.

The sum s of a summable family $(x_i)_{i \in I}$ of elements of G is usually denoted by $\sum_{i \in I} x_i$.

Proposition 1 If $(x_i)_{i \in I}$ is a summable family of elements of a Hausdorff Abelian group G having a sum s , then for any permutation σ of I , s is also the sum of the summable family $(x_{\sigma(i)})_{i \in I}$.

Proposition 2 If $(x_i)_{i \in I}$ is a summable family of elements of a Hausdorff Abelian group G , then for every neighborhood V of zero, $x_i \in V$ for all but finitely many $i \in I$.

Proposition 3 Let G be the Cartesian product of a family $(G_\lambda)_{\lambda \in L}$ of Hausdorff Abelian groups (G has the product topology). Then s is the sum of a family $(x_i)_{i \in I}$ of elements of G if, and only if, $\pi_\lambda(s)$ is the sum of $(\pi_\lambda(x_i))_{i \in I}$ for each $\lambda \in L$ (where π_λ is the canonical component from G onto G_λ).

Proposition 4 If ϕ is a continuous homomorphism from a Hausdorff Abelian group G_1 to a Hausdorff Abelian group G_2 , and if $(x_i)_{i \in I}$ is a summable family of elements of G_1 , then $(\phi(x_i))_{i \in I}$ is summable in G_2 , and $\sum_{i \in I} \phi(x_i) = \phi\left(\sum_{i \in I} x_i\right)$.

3 The statement

The objective of this paper is to prove the following result, and to observe some its consequences.

Theorem 5 Let \mathbb{K} be an Hausdorff topological field (it might be non-discrete!), and X be any set. Then, the topological dual $(\mathbb{K}^X)'$ of \mathbb{K}^X , under the product topology, is isomorphic (as a \mathbb{K} -vector space) to $\mathbb{K}^{(X)}$.

In order to illustrate the scope of this result, let us assume for an instant that \mathbb{K} is a discrete field, and that V is an Hausdorff topological \mathbb{K} -vector space. The topology on V is said to be *linear* if, and only if, it has a neighborhood basis of zero consisting of vector subspaces open in the topology.

Proposition 6 (??) Let V be a \mathbb{K} -vector space with a linear topology. Then the following conditions are equivalent:

1. V is complete, and all its open subspaces are of finite codimension.
2. V is an inverse limit of discrete finite-dimensional vector spaces, with the inverse limit topology.
3. V is isomorphic, as a topological vector space, to the algebraic dual W^* of a discrete vector space W , with the topology of simple convergence. Equivalently, V is isomorphic to \mathbb{K}^X , with the product topology, for some set X .

A topological vector space with the above equivalent properties is called *linearly compact*. This kind of spaces is not so important for this paper, but we obtain a characterization of their topological duals as a

minor consequence of our main result: we get that the topological dual of any linearly compact vector space is isomorphic to some $\mathbb{K}^{(X)}$.

We can also deduce an immediate corollary that is a partial reciprocal of our main result.

Corollary 1 *Let \mathbb{K} be a topological field, and X be a set. Let us assume that \mathbb{K}^X has the product topology. Suppose that X is infinite. Then, $(\mathbb{K}^X)'$ is isomorphic to $\mathbb{K}^{(X)}$ if, and only if, \mathbb{K} is Hausdorff.*

Proof: If \mathbb{K} is Hausdorff, then according to Theorem 5 $(\mathbb{K}^X)'$ is isomorphic to $\mathbb{K}^{(X)}$. Now, let R be an indiscrete topological ring. Then, with respect to the trivial topology on R , $(R^X)' = (R^X)^*$. Because X is infinite, the field \mathbb{K} cannot be indiscrete. But ring topologies on a field (and in particular field topologies) may be either Hausdorff or the indiscrete one (see ?). \square

4 The proof of Theorem 5

Lemma 7 *Let R be a commutative ring with unit, and X be a set. Let us define*

$$\begin{aligned} \Phi: \quad R^{(X)} &\rightarrow R^{R^X} \\ p &\mapsto \left(\begin{array}{ccc} \Phi(p): & R^X & \rightarrow R \\ f & \mapsto & \langle p, f \rangle \end{array} \right). \end{aligned} \quad (3)$$

Then, for every $p \in R^{(X)}$, $\Phi(p) \in (R^X)^$, Φ is R -linear and one-to-one.*

Proof: The first and second properties are obvious. Let $p \in R^{(X)}$ such that $\Phi(p) = 0$, then for every $f \in R^X$, $\Phi(p)(f) = 0$, and in particular, for every $x \in X$, $0 = \Phi(p)(\delta_x) = \langle p, \delta_x \rangle = p(x)$, in such a way that $p = 0$. \square

Lemma 8 *Assume that R is a topological ring (Hausdorff or not) and that R^X has the product topology. Then for every $p \in R^{(X)}$, $\Phi(p)$ is continuous, i.e., $\Phi(p) \in (R^X)'$.*

Proof: It is clear since $\Phi(p)$ is a finite sum of (scalar multiples) of projections. \square

Lemma 9 *Let us suppose that R is an Hausdorff topological ring, X is a set, and that R^X has the product topology. For every $f \in R^X$, the family $(f(x)\delta_x)_{x \in X}$ is summable with sum f .*

Proof: It is sufficient to prove that for every $x_0 \in X$ the family

$$(\langle \delta_{x_0}, f(x)\delta_x \rangle)_{x \in X} = (f(x)\delta_x(x_0))_{x \in X}$$

is summable in R with sum $\langle \delta_{x_0}, f \rangle = f(x_0)$ which is immediate. \square

Lemma 10 *Under the same assumptions as Lemma 9, if $\ell \in (R^X)'$, then*

$$Y_\ell = \{x \in X : \ell(\delta_x) \text{ is invertible in } R\}$$

is finite.

Proof: Since ℓ is continuous (and linear), and $(f(x)\delta_x)_{x \in X}$ is summable with sum f , then $(f(x)\ell(\delta_x))_{x \in X}$ is summable in R with sum $\ell(f)$ for every $f \in R^X$. Let us define

$$\begin{aligned} f: X &\rightarrow R \\ x &\mapsto \begin{cases} \ell(\delta_x)^{-1} & \text{if } x \in Y_\ell, \\ 0_R & \text{otherwise.} \end{cases} \end{aligned} \quad (4)$$

Then, $(f(x)\ell(\delta_x))_{x \in X}$ is summable with sum $\ell(f)$ in R . According to properties of summability, for every neighborhood U of 0_R in R , $f(x)\ell(\delta_x) \in U$ for all but finitely many $x \in X$. Since R is assumed Hausdorff, there is some neighborhood U of 0_R such that $1_R \notin U$. If Y is not finite, then $1_R = f(x)\ell(\delta_x) \notin U$ for every $x \in Y$ which is a contradiction. \square

Lemma 11 *Let \mathbb{K} be an Hausdorff topological field, X be a set, and assume that \mathbb{K}^X is equipped with the product topology. Let $\ell \in (\mathbb{K}^X)^*$. If $\ell \in (\mathbb{K}^X)'$, then $\ell(\delta_x) = 0$ for all but finitely many $x \in X$.*

Proof: According to Lemma 10, the set $\{x \in X : \ell(\delta_x) \text{ is invertible in } \mathbb{K}\} = \{x \in X : \ell(\delta_x) \neq 0\}$ is finite. \square

Lemma 12 *Under the same assumptions as Lemma 11, Φ is onto.*

Proof: Let $\ell \in (\mathbb{K}^X)'$. Let us define

$$\begin{aligned} p_\ell: X &\rightarrow \mathbb{K} \\ x &\mapsto \ell(\delta_x). \end{aligned} \quad (5)$$

A priori $p_\ell \in \mathbb{K}^X$. But according to Lemma 11, actually $p_\ell \in \mathbb{K}^{(X)}$. Let $f \in \mathbb{K}^X$. We have

$$\Phi(p_\ell)(f) = \langle p_\ell, f \rangle = \sum_{x \in X} p_\ell(x)f(x) = \sum_{x \in X} \ell(\delta_x)f(x) = \ell(f)$$

and then $\Phi(p_\ell) = \ell$. \square

Now it is easy to conclude the proof of Theorem 5 since it follows directly from Lemmas 7 and 12.

Remark 1 *1. The algebraic dual of \mathbb{K}^X may be distinct from $\mathbb{K}^{(X)}$. Indeed, let $(e_i)_{i \in I}$ be an algebraic basis of \mathbb{K}^X (the existence of such a basis requires the axiom of choice for sets X of arbitrary large cardinal number). Therefore, every map $f \in \mathbb{K}^X$ may be (uniquely) written as a finite linear combination $\sum_{i \in I} f_i e_i$, with $f_i \in \mathbb{K}$ for each $i \in I$. Let us consider the map $\ell: \mathbb{K}^X \rightarrow \mathbb{K}$ such that*

$$\ell(f) = \sum_{i \in I} f_i. \text{ Clearly, } \ell \text{ is a linear form, that is, an element of the algebraic dual } (\mathbb{K}^X)^* \text{ of } \mathbb{K}^X.$$

The family $(\delta_x)_{x \in X}$ is linearly independent in \mathbb{K}^X . Thus we may consider the algebraic basis of \mathbb{K}^X that extends $(\delta_x)_{x \in X}$. Now, the corresponding functional ℓ has a nonzero value for each δ_x . Therefore, if X is infinite, then ℓ does not belong to the image of Φ , or, in other terms, $\ell \notin (\mathbb{K}^X)'$. In particular, whenever \mathbb{K} is an Hausdorff topological field, \mathbb{K}^X has the product topology, and X is infinite, then ℓ is discontinuous at zero (and thus on the whole \mathbb{K}^X).

2. A field topology may be either Hausdorff or the indiscrete one. It seems to remain too little choice. Actually it is known (see ??) that every infinite field \mathbb{K} has $2^{2^{|\mathbb{K}|}}$ non-homeomorphic topologies (where $|X|$ denotes the cardinal number of a set X).
3. Let R be a discrete ring, and X be a set. Let X^* be the free monoid on the alphabet X , ϵ be the empty word, and $|\omega|$ be the length of a word $\omega \in X^*$. Let us define $\mathfrak{M}_{\geq n} = \{f \in R^X : \nu(f) \geq n\}$, $n \in \mathbb{N}$, where $\nu(f) = \inf\{n \in \mathbb{N} : \exists \omega \in X^*, |\omega| \neq n, \text{ and } f(\omega) \neq 0\}$ for every non-zero $f \in R^X$ (the infimum being taken on $\mathbb{N} \cup \{\infty\}$, with $\infty > n$ for every $n \in \mathbb{N}$, then $\nu(0) = \infty$). The set R^X , seen as the R -algebra $R\langle\langle X \rangle\rangle$ of formal power series in noncommutative variables, may be topologized (as a topological R -algebra – see ? – and, therefore, as a topological R -module) by the decreasing filtration of ideals $\mathfrak{M}_{\geq n}$: this is an example of the so-called Krull topology (see ?), and it is the usual topology considered for formal power series in combinatorics and algebra; in case X is reduced to an element x , we recover the usual \mathfrak{M} -adic topology of $\mathbb{K}[[x]]$, where $\mathfrak{M} = \langle x \rangle$ is the principal ideal generated by x . Whenever X is finite, this Krull topology coincides⁽ⁱⁱⁱ⁾ with the product topology with a discrete R . According to Theorem 5, in case where X is finite and R is a topological field \mathbb{K} , the topological dual of $\mathbb{K}\langle\langle X \rangle\rangle$ (which is also a linearly compact space) is the space of polynomials $\mathbb{K}\langle X \rangle$ in noncommutative variables.
4. Take any monoid with a zero (see ?) with the finite decomposition property (see ??), and let R be a ring. Let us consider the total contracted R -algebra $R_0[[M]]$ of the monoid with zero M (see ?) that consists, as a R -module, of $\{f \in R^M : f(0_M) = 0_R\}$, where 0_M is the zero of M , while 0_R is the zero of the ring R . It is clear that $R_0[[M]] \cong R^{M^*}$ (as R -algebras), with $M^* = M \setminus \{0_M\}$. Now, let us assume that \mathbb{K} is an Hausdorff topological field. The product topology on \mathbb{K}^M induces the product topology on \mathbb{K}^{M^*} , that corresponds to that of $\mathbb{K}_0[[M]]$. The topological dual of \mathbb{K}^{M^*} being $\mathbb{K}^{(M^*)}$ it is easy to check that $(\mathbb{K}_0[[M]])'$ is isomorphic to the (usual) contracted algebra (see ?) $\mathbb{K}_0[M]$ of the monoid with zero M .
5. Theorem 5 may be applied for the discrete topology on $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, but also for the usual topologies of \mathbb{R} and \mathbb{C} , in such a way that the topological dual spaces $(\mathbb{R}^X)'$ or $(\mathbb{C}^X)'$ for both discrete topology and the topology induced by the (usual) absolute values on \mathbb{R}, \mathbb{C} are identical since isomorphic to $\mathbb{R}^{(X)}$ or $\mathbb{C}^{(X)}$. Notice that \mathbb{K}^X is a Fréchet space (see ?), real or complex depending on whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, when is considered the product topology relative to the absolute value, and as such allows functional analysis like, for instance, Banach-Steinhaus, open map and closed graph theorems that do not hold in the case of the same space with the product topology relative to a discrete \mathbb{K} .

5 Consequence on continuous endomorphisms

As explained in the Introduction, the rigidity of the dual space with respect to the change of product topologies forces continuous linear maps (with respect to any of those topologies) to be represented by “row-finite” matrices.

Let \mathbb{K} be an Hausdorff topological field, and X, Y be two sets. We suppose that \mathbb{K}^Z has the product topology for $Z \in \{X, Y\}$. The set of all linear maps (resp. continuous linear maps) from \mathbb{K}^X to \mathbb{K}^Y is

⁽ⁱⁱⁱ⁾ Notice that if X is infinite both topologies are distinct: for instance, let $(x_n)_{n \geq 0}$ be a sequence of distinct elements of X , then this family is easily seen summable in the product topology with R discrete, while it does not converge in the Krull topology.

denoted by $\text{Hom}_{\mathbb{K}}(\mathbb{K}^X, \mathbb{K}^Y)$ (resp. $\mathcal{L}(\mathbb{K}^X, \mathbb{K}^Y)$). We denote by $\mathbb{K}^{Y \times (X)}$ the set of all maps $M: Y \times X \rightarrow \mathbb{K}$ such that for each $y \in Y$, the set $\{x \in X: M(y, x) \neq 0\}$ is finite. Recall that if $p \in \mathbb{K}^{(X)}$, then its support is given by

$$\text{supp}(p) = \{x \in X: p(x) \neq 0\}. \quad (6)$$

Let $\phi \in \mathcal{L}(\mathbb{K}^X, \mathbb{K}^Y)$. We define the following map:

$$\begin{aligned} M_\phi: \quad Y \times X &\rightarrow \mathbb{K} \\ (y, x) &\mapsto \langle \delta_y, \phi(\delta_x) \rangle. \end{aligned} \quad (7)$$

Lemma 13 *For each $\phi \in \mathcal{L}(\mathbb{K}^X, \mathbb{K}^Y)$, $M_\phi \in \mathbb{K}^{Y \times (X)}$, and the map $\phi \mapsto M_\phi$ is into.*

Proof: For every $x \in X$, the map

$$\begin{aligned} \mathbb{K}^X &\rightarrow \mathbb{K} \\ f &\mapsto \langle \delta_y, \phi(f) \rangle \end{aligned} \quad (8)$$

is an element of $(\mathbb{K}^X)'$ (because it is the composition of ϕ and the projection onto $\mathbb{K}\delta_y$). According to Theorem 5, there is one and only one $p_{\phi,y} \in \mathbb{K}^{(X)}$ such that for every $f \in \mathbb{K}^X$, $\langle p_{\phi,y}, f \rangle = \langle \delta_y, \phi(f) \rangle$. In particular, for every $x \in X$,

$$\begin{cases} p_{\phi,y}(x) & \text{if } x \in \text{supp}(p_{\phi,y}) \\ 0_R & \text{otherwise} \end{cases} = \sum_{z \in X} p_{\phi,y}(z) \delta_x(z) = \langle p_{\phi,y}, \delta_x \rangle = \langle \delta_y, \phi(\delta_x) \rangle. \quad (9)$$

Therefore $\{x \in X \mid \langle \delta_y, \phi(\delta_x) \rangle \neq 0\} = \text{supp}(p_{\phi,y})$, and then $M_\phi \in \mathbb{K}^{Y \times (X)}$.

Suppose that $M_\phi = M_{\phi'}$, then for every $(y, x) \in Y \times X$, $\langle \delta_y, \phi(\delta_x) \rangle = \langle \delta_y, \phi'(\delta_x) \rangle$. Then, by bilinearity, $\phi(\delta_x)(y) - \phi'(\delta_x)(y) = \langle \delta_y, \phi(\delta_x) - \phi'(\delta_x) \rangle = 0$. Since this last equality holds for every $y \in Y$, $\phi(\delta_x) = \phi'(\delta_x)$ for every $x \in X$. Now, let $f \in \mathbb{K}^X$, since $f = \sum_{x \in X} f(x) \delta_x$ (sum of a summable family), by continuity, $\phi(f) = \sum_{x \in X} f(x) \phi(\delta_x) = \sum_{x \in X} f(x) \phi'(\delta_x) = \phi'(f)$. \square

Theorem 14 *The sets $\mathcal{L}(\mathbb{K}^X, \mathbb{K}^Y)$ and $\mathbb{K}^{Y \times (X)}$ are equipotent. More precisely the map $\phi \mapsto M_\phi$ of Lemma 13 is onto.*

Proof: Let $M \in \mathbb{K}^{Y \times (X)}$. Let us define $\psi_M: \mathbb{K}^X \rightarrow \mathbb{K}^Y$ by

$$\psi_M(f) = \psi_M\left(\sum_{x \in X} f(x) \delta_x\right) = \sum_{y \in Y} \left(\sum_{x \in X} M(y, x) f(x)\right) \delta_y.$$

(Clearly the second sum on $x \in X$ has only finitely many non zero terms since $M \in \mathbb{K}^{Y \times (X)}$, and therefore is defined in \mathbb{K} .) The map ψ_M is linear. To see this, since $\psi_M(f) \in \mathbb{K}^Y$, it is sufficient to prove

that for every $\lambda \in \mathbb{K}$, $f, g \in \mathbb{K}^X$, and every $y \in Y$, $\psi_M(\lambda f + g)(y) = \lambda \psi_M(f)(y) + \psi_M(g)(y)$. This equality to prove is equivalent to the following:

$$\begin{aligned}
& \psi_M(\lambda f + g)(y) &= \lambda \psi_M(f)(y) + \psi_M(g)(y) \\
\Leftrightarrow \langle \delta_y, \psi_M(\lambda f + g) \rangle &= \lambda \langle \delta_y, \psi_M(f) \rangle + \langle \delta_y, \psi_M(g) \rangle \\
\Leftrightarrow \langle \delta_y, \psi_M(\lambda f + g) \rangle &= \langle \delta_y, \lambda \psi_M(f) + \psi_M(g) \rangle \\
\Leftrightarrow \sum_{x \in X} M(y, x)(\lambda f(x) + g(x)) &= \sum_{x \in X} (\lambda M(y, x)f(x) + M(y, x)g(x)) .
\end{aligned} \tag{10}$$

But the last equality is obvious (since the sums have only finitely many non zero terms). Let us prove that ψ_M is continuous. It is sufficient to prove that for every $y \in Y$, $\ell_{M,y}: \mathbb{K}^X \rightarrow \mathbb{K}$, defined by $\ell_{M,y}(f) = \langle \delta_y, \psi_M(f) \rangle = \sum_{x \in X} M(y, x)f(x)$, is continuous. This is the case since $\ell_{M,y}$ is a finite sum of scalar multiples of projections. Therefore $\psi_M \in \mathcal{L}(\mathbb{K}^X)$. Finally we prove that $M_{\psi_M} = M$. Let $(y, x) \in Y \times X$, we have

$$\begin{aligned}
M_{\psi_M}(y, x) &= \langle \delta_y, \psi_M(\delta_x) \rangle \\
&= \langle \delta_y, \sum_{y' \in Y} \left(\sum_{z \in X} M(y', z)\delta_x(z) \right) \delta_{y'} \rangle \\
&= \sum_{z \in X} M(y, z)\delta_x(z) \\
&= M(y, x) .
\end{aligned} \tag{11}$$

The map $\phi \mapsto M_\phi$ is thus onto, and, by Lemma 13, it is a bijection. \square

Note 1 1. If $X = Y$, then denoting the set $\mathcal{L}(\mathbb{K}^X, \mathbb{K}^X)$ of all continuous endomorphisms by $\mathcal{L}(\mathbb{K}^X)$, we find that $\mathcal{L}(\mathbb{K}^X)$ and $\mathbb{K}^{X \times (X)}$ are equipotent.

2. If $X = Y = \mathbb{N}$, then is recovered the usual notion of row-finite matrices (see ?).

3. If Y is reduced to a single element x , then $\mathcal{L}(\mathbb{K}^X) = \mathcal{L}(\mathbb{K}^X, \mathbb{K}) \cong \mathcal{L}(\mathbb{K}^X, \mathbb{K}^{\{x\}}) \cong \mathbb{K}^{\{x\} \times (X)} \cong \mathbb{K}^{(X)}$, which is our Theorem 5.

6 Topological duality and completion

In this section, we construct explicitly the canonical isomorphism between the topological duals of $\mathbb{K}^{(X)}$ and \mathbb{K}^X using the fact that the former is the completion of the later.

Recall the following definition: let \mathbb{K} be a field with an Hausdorff (field) topology, and V be an Hausdorff topological \mathbb{K} -vector space. The *completion* of V is a pair (\widehat{V}, i) where \widehat{V} is complete (Hausdorff) topological \mathbb{K} -vector space and $i: V \rightarrow \widehat{V}$ such that

1. The map i is an isomorphism of topological \mathbb{K} -vector space structures from V into \widehat{V} , i.e., i is both an (algebraic) isomorphism and a homeomorphism into \widehat{V} (or in other terms, i is a bicontinuous one-to-one R -linear map, or i is a continuous one-to-one R -linear map and its inverse $i^{-1}: i(V) \rightarrow V$ is continuous where $i(V)$ has the subspace topology induced by \widehat{V});

2. The image $i(V)$ is dense in \widehat{V} ;
3. For any complete (Hausdorff) \mathbb{K} -vector space W and any continuous and linear map $\phi: V \rightarrow W$, there exists one and only one map $\widehat{\phi}: \widehat{V} \rightarrow W$ such that $\widehat{\phi} \circ i = \phi$.

Remark 2 According to ?, \mathbb{K}^X is complete and separated (with respect to the product topology) if, and only if, \mathbb{K} is itself a complete Hausdorff field. In particular, the topological dual of \mathbb{K}^X does not depend on whether or not \mathbb{K} is a complete field.

Now let us assume that \mathbb{K} has the discrete topology (therefore \mathbb{K} is a complete Hausdorff topological field). Clearly, \mathbb{K}^X is the completion of $\mathbb{K}^{(X)}$ (the latter being equipped with the initial topology with respect to the obvious projections which coincides with the subspace topology induced by \mathbb{K}^X for its product topology). According to the definition of a completion, taking \mathbb{K} in place of W , it is clear that there exists a canonical isomorphism Ψ between the topological dual of $\mathbb{K}^{(X)}$ and \mathbb{K}^X . The isomorphism

$$\begin{array}{ccc} \Psi: & (\mathbb{K}^{(X)})' & \rightarrow & (\mathbb{K}^X)' \\ & \ell & \mapsto & \widehat{\ell} \end{array} \quad (12)$$

has inverse $\Psi^{-1}(\ell) = \ell \circ i = \ell|_{\mathbb{K}^{(X)}}$ for $\ell \in (\mathbb{K}^X)'$. The isomorphism Ψ may be given a precise definition. Let $\ell \in (\mathbb{K}^{(X)})'$. Then we have

$$\begin{array}{ccc} \Psi(\ell) = \widehat{\ell}: & \mathbb{K}^X & \rightarrow & \mathbb{K} \\ & f & \mapsto & \sum_{x \in X} f(x)\ell(\delta_x). \end{array} \quad (13)$$

Indeed since $f = \sum_{x \in X} f(x)\delta_x$ (sum of a summable family), we have

$$\begin{aligned} \Psi(\ell)(f) &= \widehat{\ell}(f) \\ &= \widehat{\ell}\left(\sum_{x \in X} f(x)\delta_x\right) \\ &= \sum_{x \in X} f(x)\widehat{\ell}(\delta_x) \\ &\quad \text{(since } \widehat{\ell} \text{ is continuous)} \\ &= \sum_{x \in X} f(x)\ell(\delta_x). \\ &\quad \text{(since } \delta_x \in \mathbb{K}^{(X)}) \end{aligned} \quad (14)$$

According to previously introduced notations, $\Psi(\ell)(f) = \widehat{\ell}(f) = \langle p_{\Psi(\ell)}, f \rangle$, where we recall that $p_{\Psi(\ell)} \in \mathbb{K}^{(X)}$ is defined by $p_{\Psi(\ell)}(x) = \Psi(\ell)(\delta_x) = \widehat{\ell}(\delta_x) = \ell(\delta_x)$ (since $\delta_x \in \mathbb{K}^{(X)}$). Note that $p_{\Psi(\ell)} = \Phi^{-1}(\Psi(\ell)) = \Phi^{-1}(\widehat{\ell})$. The map $\Phi^{-1} \circ \Psi: (\mathbb{K}^{(X)})' \rightarrow \mathbb{K}^{(X)}$ is an isomorphism (composition of isomorphisms). The polynomial $\Phi^{-1}(\Psi(\ell)) \in \mathbb{K}^{(X)}$ for $\ell \in (\mathbb{K}^{(X)})'$ is thus given by $\Phi^{-1}(\Psi(\ell))(x) = \ell(\delta_x)$ for $x \in X$. We can check that $\ell(p) = \langle \Phi^{-1}(\Psi(\ell)), i(p) \rangle$ for any $\ell \in (\mathbb{K}^{(X)})'$, and $p \in \mathbb{K}^{(X)}$. Indeed, $\ell(p) = \Psi(\ell)(i(p)) = \langle \Phi^{-1}(\Psi(\ell)), i(p) \rangle$.

7 Weak topology

Let \mathbb{K} be a topological field, (V, τ) be a topological \mathbb{K} -vector space, and V' be its topological dual. We call V' -weak topology the weakest topology on V such that the elements of V' are continuous. Let us denote by τ_w this topology. Since the elements of V' are continuous (for τ), we have $\tau_w \subseteq \tau$. It can be shown that (V, τ_w) is also a \mathbb{K} -topological vector space, separated if \mathbb{K} is so and V' separates the elements of V . The topological dual V'_w of (V, τ_w) is called *weak dual* of V .

Corollary 2 *Let (\mathbb{K}, τ) be an Hausdorff topological field, X be a set, and \mathbb{K}^X endowed with the product topology, denoted by $\pi(X, \tau)$, of $|X|$ copies of (\mathbb{K}, τ) . The weak dual $(\mathbb{K}^X)'_w$ of (\mathbb{K}^X, τ_w) is (isomorphic to) $\mathbb{K}^{(X)}$.*

Proof: An element of $\mathbb{K}^{(X)}$ is obviously a continuous linear form (via the isomorphism between $(\mathbb{K}^X)'$ and $\mathbb{K}^{(X)}$ of Theorem 5) by definition of the $\mathbb{K}^{(X)}$ -weak topology. Therefore, $\mathbb{K}^{(X)} \subseteq (\mathbb{K}^X)'_w$. Let us prove that the product and the weak topologies are actually equal. The weak topology τ_w is, by definition, the weakest topology for which every linear form $\langle p, \cdot \rangle: \mathbb{K}^X \rightarrow (\mathbb{K}, \tau)$ is continuous ($p \in \mathbb{K}^{(X)}$). In particular, the projections $\langle \delta_x, \cdot \rangle$ are also continuous. Therefore τ_w is stronger than $\pi(X, \tau)$ (since the later is the weakest topology with this property). So $\pi(X, \tau) \subseteq \tau_w$. Let us prove the reciprocal inclusion. To do so, it is sufficient to prove that the identity map $\text{id}: (\mathbb{K}^X, \pi(X, \tau)) \rightarrow (\mathbb{K}^X, \tau_w)$ is continuous, which is equivalent (according to usual properties of initial topology) to the fact that for every $p \in \mathbb{K}^{(X)}$, the map

$$\begin{array}{ccc} \langle p, \cdot \rangle: & (\mathbb{K}^X, \pi(X, \tau)) & \rightarrow (\mathbb{K}, \tau) \\ & f & \mapsto \langle p, f \rangle \end{array} \quad (15)$$

is continuous, which is obviously the case since these maps are sum of a finite number of (scalar multiples of) projections. Therefore a linear form $\ell: \mathbb{K}^X \rightarrow (\mathbb{K}, \tau)$ is continuous with respect to the weak topology if, and only if, it is continuous with respect to the product topology, and therefore is an element of $\mathbb{K}^{(X)}$ (according to Theorem 5). \square

